

8.3 Notes: Geometric Sequences and Series

A sequence is geometric if the ratios of consecutive terms are the same. So

$$a_1, a_2, a_3, \dots, a_n \text{ is geometric if there is a number } r \text{ such that } \frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \dots = r : r \neq 0.$$

Determine whether each sequence is geometric. If so, find r .

<p>1. 3, -9, 27, -81, 243...</p> <p>$\frac{-9}{3} = -3$, $\frac{27}{-9} = -3$ Yes! $r = -3$</p>	<p>2. $\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \dots$</p> <p>Yes! $r = \frac{1}{4}$</p>	<p>3. 1, 2, 6, 24, 120...</p> <p>NO</p>
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Fill in the blanks with the missing terms of the geometric sequence.

<p>4. 144, <u>72</u>, 36, <u>18</u>, <u>9</u>, 4.5</p> <p>$\frac{4.5}{9} = \frac{1}{2} \therefore r = \frac{1}{2}$</p>		
<p>5. 5, 15, 45, _____, _____</p> <p>Part 1: $a_2 = 5 \cdot 3$ $a_3 = 5 \cdot 3 \cdot 3$ $a_4 = 5 \cdot 3 \cdot 3 \cdot 3$</p>	<p>Part 2: $a_2 = a_1 \cdot 3^1$ $a_3 = a_1 \cdot 3^2$ $a_4 = a_1 \cdot 3^3$ $a_n = a_1 \cdot r^{(n-1)}$</p>	<p>Part 3: Find a_{10} $a_{10} = 5(3)^9$</p>

In a geometric sequence, we repeatedly multiply by r to get successive terms. Hence, to write an explicit formula for the n^{th} term, we must have a power of r .

$$a_n = a_1 \cdot r^{(n-1)}$$

Examples

<p>6. Find the n^{th} term of the geometric sequence. Write the first five terms of the geometric sequence whose first term is 2 and whose common ratio is 4.</p> <p>$a_1 = 2$ $r = 4$ $a_n = a_1 \cdot r^{(n-1)}$ $a_n = 2(4)^{(n-1)}$</p> <p>2, 8, 32, 128, 512</p>	<p>7. Find the 9th term of the geometric sequence whose first term is 4 and whose common ratio is $\frac{1}{2}$.</p> <p>$a_1 = 4$ $r = \frac{1}{2}$ $a_n = 4\left(\frac{1}{2}\right)^{n-1}$ $a_9 = 4\left(\frac{1}{2}\right)^8$ $a_9 = 4\left(\frac{1}{256}\right) = \frac{1}{64}$</p>	<p>8. Find the n^{th} term of the geometric sequence and then find the value of the tenth term.</p> <p>6, -2, $\frac{2}{3}, \dots$ $r = -\frac{1}{3}$ $a_1 = 6$ $a_n = 6\left(-\frac{1}{3}\right)^{(n-1)}$ $a_{10} = 6\left(-\frac{1}{3}\right)^9$ $= -\frac{2}{6561}$</p>
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Lesson #3 Geometric Sequences Practice:

key

The formula for the n^{th} term of a GEOMETRIC SEQUENCE:

$$a_n = a_1 \cdot r^{(n-1)}$$

1) Find the next 3 terms of each geometric sequence. Then write the general term, a_n .

a) -15, 30, -60, ... $a_n = -15(-2)^{(n-1)}$ $a_n =$ _____

$a_1 = -15$

$r = -2$

-15, 30, -60, 120, -240, 480

b) 10, 2, $\frac{2}{5}$, ... $\frac{2}{25}, \frac{2}{125}, \frac{1}{300}$ $a_n = 10 \left(\frac{1}{5}\right)^{(n-1)}$

$a_1 = 10$

$r = \frac{1}{5}$

2) Find the specified terms of each geometric sequence and write the rule for the n^{th} term

a) $a_1 = 7, r = -3, a_7 =$ **5103**

$a_n = a_1 \cdot r^{(n-1)}$

$a_7 = 7 \cdot (-3)^{7-1}$

$a_7 = 7(-3)^6$

Rule: $a_n = 7(-3)^{n-1}$

b) $a_1 = 144, r = \frac{1}{4}, a_5 =$ **9/16**

$a_n = a_1 \left(\frac{1}{4}\right)^{n-1}$

$a_n = 144 \left(\frac{1}{4}\right)^{n-1}$

$a_5 = 144 \left(\frac{1}{4}\right)^4 =$

Rule: $a_n = (144) \left(\frac{1}{4}\right)^{(n-1)}$

$a_5 = 144 \left(\frac{1}{4}\right)^4$
 $= \frac{3 \cdot 3}{4 \cdot 4} = \frac{9}{16} \cdot 12 \cdot 12 \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}$

c) $a_9 = 6561, r = 3, a_3 = 9$

$(9, 6561) r = 3$
 $a_n = a_1 \cdot r^{(n-1)}$
 $6561 = a_1 (3)^8$
 $1 = a_1$

$a_3 = 9$

Rule: $a_n = 1(3)^{n-1}$

d) $a_7 = 3645, r = 3, a_2 = 15$

$(7, 3645) r = 3$
 $a_n = a_1 \cdot r^{(n-1)}$
 $3645 = a_1 (3)^6$
 $3645 = a_1 (3)^6$
 $3645 = 729 a_1$
 $\frac{3645}{729} = 5 = a_1$

Rule: $a_n = 5(3)^{n-1}$

3) Find a_1 and r for each geometric sequence. Then write the rule for the n th term.

a) $a_2 = 10, a_5 = 160, a_1 = \underline{\hspace{2cm}}, r = \underline{\hspace{2cm}}$

$a_2 = a_1 \cdot r^{n-1}$
 $10 = a_1 \cdot r$
 $\frac{10}{r} = a_1$
 $160 = a_1 \cdot r^4$
 $160 = \frac{10}{r} \cdot r^5$
 $160 = 10r^4$
 $16 = r^4$
 $\pm 2 = r$
 $a_1 = \frac{10}{\pm 2} = \pm 5$

Rule: $a_n = 5(2)^{n-1}$
 or $a_n = -5(-2)^{n-1}$

b) $a_3 = 45, a_5 = 405, a_1 = \underline{\hspace{2cm}}, r = \underline{\hspace{2cm}}$

$45 = a_1 \cdot r^2$
 $405 = a_1 \cdot r^4$
 $a_1 = \frac{45}{r^2}$
 $405 = \frac{45}{r^2} \cdot r^4$
 $405 = 45r^2$
 $9 = r^2$
 $\pm 3 = r$

Rule: $a_n = 5(3)^{n-1}$
 or $a_n = 5(-3)^{n-1}$

c) $a_5 = 16, a_8 = 2, a_1 = \underline{\hspace{2cm}}, r = \underline{\hspace{2cm}}$

$16 = a_1 \cdot r^4$
 $2 = a_1 \cdot r^7$
 $a_1 = \frac{16}{r^4}$
 $2 = \frac{16}{r^4} \cdot r^7$
 $2 = 16r^3$
 $\frac{1}{8} = r^3$
 $r = \frac{1}{2}$
 $a_1 = 16 \div \frac{1}{16} = 256$
 Rule: $a_n = 256 \left(\frac{1}{2}\right)^{(n-1)}$

d) $a_7 = 6, a_{11} = \frac{3}{8}, a_1 = \underline{\hspace{2cm}}, r = \underline{\hspace{2cm}}$

$6 = a_1 \cdot r^6$
 $\frac{3}{8} = a_1 \cdot r^{10}$
 $a_1 = \frac{6}{r^6}$
 $\frac{3}{8} = \frac{6}{r^6} \cdot r^{10}$
 $\frac{3}{8} = 6r^4$
 $\frac{1}{16} = r^4$
 $r = \pm \frac{1}{2}$
 $6 \div \frac{1}{64} = a_1$
 $384 = a_1$
 Rule: $a_n = 384 \left(\pm \frac{1}{2}\right)^{n-1}$

Lesson #4 Geometric Series

Key

The sum of the first n terms of a Geometric Series can be found by using the formula:

$$S_n = \frac{a_1(1-r^n)}{1-r}, \text{ where } r \neq 1$$

Example 1: Find each indicated sum for each geometric series.

a) $a_1 = 256, r = -\frac{1}{3}, S_8 = \frac{15,616}{81}$

b) $a_1 = 4, r = 0.5; S_8 = 7.96875$

$$S_8 = \frac{4(1-.5^8)}{1-.5}$$

$$= \frac{4(1-.5^8)}{.5}$$

e) $3 + 12 + 48 + \dots; S_6 = 4095$

f) $0.2 + 1 + 5 + 25 + \dots; S_8 = 19,531.2$
 $r = 5$
 $n = 8$
 $a_1 = .2$

$$S_8 = \frac{.2(1-5^8)}{1-5}$$

$$= \frac{2}{10}(-390624)$$

$$= \frac{-390624}{5}$$

$$19531.2$$

Example 2: For each geometric series, find a_1 .

a) $r = 4, S_5 = 2730 = 2$

b) $r = 3, S_8 = 22,960 = a_1 = 7$

$S_8 =$
 $22,960 = a_1 \frac{(1-3^8)}{1-3}$

c) $r = -\frac{1}{2}, S_7 = \frac{215}{64} = 5$

Find a_1
 $-45920 = a_1(1-3^8)$
 $= a_1(-6560)$
 $7 = a_1$

Example 3: For A, Rewrite the sum in expanded form. Then find each sum using your method of choice.

a) $\sum_{n=1}^5 4(3)^{n-1} = \underline{484}$

b) $\sum_{n=1}^7 81\left(\frac{1}{3}\right)^{n-1} = \underline{\hspace{2cm}}$

$4 + 12 + 48 + 192 + 768$

$S_5 = \frac{4(1-3^5)}{1-3} = 484$

c) $\sum_{n=1}^6 5(-2)^{n-1} = \underline{-105}$

d) $\sum_{n=1}^7 256\left(\frac{1}{4}\right)^{n-1} = \underline{\hspace{2cm}}$

Recall \leftarrow # term
 Σ Rule
 $n =$

Use $a_n = a_1 r^{(n-1)}$ to get Rule.

Example 4: Express in sigma notation the sum of the first eight terms of each geometric series.

a) $5 + (-15) + 45 + \dots$

$\sum_{n=1}^8 5(-3)^{(n-1)}$

b) $192 + 96 + 48 + \dots$

$a_1 = 192$
 $r = \frac{1}{2}$
 $n = 8$
 $\sum_{n=1}^8 192\left(\frac{1}{2}\right)^{(n-1)}$

c) $4 + (-12) + 36 + \dots$
 $a_1 = 4$
 $r = -3$
 $n = 8$
 S_8
 $\sum_{n=1}^8 4(-3)^{(n-1)}$

d) $240 + 360 + 540 + \dots$
 $a_1 = 240$
 $r = 1.5$
 $n = 8$
 $\sum_{n=1}^8 240(1.5)^{(n-1)}$

$\sum_{n=1}^8$

so $\sum_{n=1}^8 4(-3)^{(n-1)}$

7. Find the indicated partial-sum for each arithmetic series.

a) $a_1 = 14, a_{15} = -283, S_{15} = \underline{-2017.5}$

$$S_{15} = \frac{15(14 + (-283))}{2} = \frac{15(-269)}{2} = -2017.5$$

c) $4 + 11 + 18 + \dots + 172 = \underline{2200}$
 $d = 7$ $172 = 4 + (n-1) \cdot 7$
 $a_1 = 4$ $168 = 7n - 7$
 $a_n = 172$ $175 = 7n$ $n = 25$

b) $8 + 3 + (-2) + (-7) + \dots S_{13} = \underline{-286}$
 $a_1 = 8$ $n = 13$ $d = -5$
 $a_{13} = 8 + (12)(-5)$
 $a_{13} = -52$

$$S_{13} = \frac{n(a_1 + a_n)}{2}$$

$$S_{13} = \frac{13(8 + (-52))}{2} = -572$$

8. Find the indicated partial sum for each geometric series.

a) $a_1 = 2, r = 3; S_9 = \underline{19682}$ $S_{25} = \frac{25(4 + 172)}{2}$

b) $0.25 + 1 + 4 + \dots; S_8 = \underline{-572}$
 $r = 4$ $a_1 = .25$

$$S = \frac{a_1(1-r^n)}{1-r}$$

$$S_9 = \frac{2(1-3^9)}{1-3} = -1(1-3^9)$$

$$S_8 = \frac{.25(1-4^8)}{1-4} = 5461.25$$

9. For each geometric series, find a_1 .

a) $r = -3, S_9 = 34447$ $a_1 = \underline{7}$
 $n = 9$
 $34447 = a_1 \frac{(1 - (-3)^9)}{4}$
 $137788 = 19684 a_1$
 $a_1 = 7$

b) $r = \frac{1}{2}, S_5 = -\frac{93}{16}$ $a_1 = \underline{-3}$
 $S_5 = -\frac{93}{16} = a_1 \frac{(1 - (\frac{1}{2})^5)}{-1/2}$
 $-\frac{93}{16} = a_1 (-3)$
 $a_1 = -3$

10. Write the arithmetic series in expanded form, and then find its sum.

AP a) $\sum_{n=1}^7 (8n-5) = \underline{189}$
 $3 + 11 + 19 + 27 + 35 + 43 + 51$
 $S_7 = \frac{7(3+51)}{2} = 7 \cdot 27 = \underline{189}$

b) $\sum_{k=12}^{17} (3k+4) = \underline{285}$ *Arith.*
 $\frac{12}{13}$ $40 + 43 + 46 + 49 + 52 + 55$
 $S_6 = \frac{6(40+55)}{2} = 3(95) = \underline{285}$

11. Write the geometric series in expanded form, and then find its sum.

a) $\sum_{n=1}^8 \frac{3}{5}(5^{n-1}) = \underline{1, 5, 25, 125, \dots, 78125}$
 $a_1 = 1$ $r = 5$
 $a_8 = 78125$ $S_n = \frac{a_1(1-r^n)}{1-r}$
 $n = 8$
 $S_8 = \frac{1(1-5^8)}{1-5} = \underline{97656}$

b) $\sum_{k=2}^7 -3^{(k-1)} = \underline{-3 + -9 + -27 + \dots + -729}$
 $r = 3$
 $a_1 = -3$ $a_6 = -729$
 $S_6 = \frac{-3(1-3^6)}{1-3} = \underline{-1092}$

12. Express in sigma notation the sum of the first six terms for each.

a) $7+10+13+\dots =$ _____

$$\sum_{n=1}^6 (3n+4)$$

$d=3$
 $a_1=7$
 $a_n = a_1 + (n-1)d$
 $a_n = 7 + 3n - 3$
 $a_n = 3n + 4$

b) $\frac{3}{32} + \frac{3}{4} + 6 + \dots =$ _____

$r=8$ $a_1 = \frac{3}{32}$
 $a_n = \frac{3}{32} \cdot 8^{(n-1)}$
 $\sum_{n=1}^6 \left[\frac{3}{32} \cdot 8^{(n-1)} \right]$

c) $(-81) + (-27) + (-9) + \dots =$ _____

$r = \frac{1}{3}$
 $a_1 = -81$
 $\sum_{n=1}^6 \left[-81 \left(\frac{1}{3} \right)^{(n-1)} \right]$

$a_n = a_1 \cdot r^{n-1}$
 $a_n = -81 \left(\frac{1}{3} \right)^{n-1}$

d) $1.2 + 2.6 + 4 + 5.4 + \dots =$ _____

$d=1.4$ $a_1=1.2$
 $\sum_{n=1}^6 [1.2 + (n-1) \cdot 1.4]$

13. As shown in the table, the cost of renting a car is related to the number of miles driven. If a rental car is driven 600 miles, what is the cost?

Rental Car Cost					
Miles	100	200	300	400	500
Cost	\$44.75	\$53.25	\$61.75	\$70.25	\$78.75

\$ 87.25

$d = 8.50$
 $78.75 + 8.50 = 87.25$

14. Donna is employed by a doctor. She placed a sample of bacteria in a culture dish and recorded the number of bacteria present each 30 minutes beginning at 12:00 PM. The table shows Donna's data. If the pattern of bacterial growth remains constant, how many bacteria should be present in the culture dish at 3:00 PM?

Bacterial Growth			
Time	12:00 PM	12:30 PM	1:00 PM
Bacteria Present	120	360	1080

let $a_1 = 12$ p.m.
 $a_2 = 12:30$ p.m.
 $a_7 = 3$ p.m.

$r=3$
 $a_n = a_1 \cdot r^{(n-1)}$
 $a_7 = 120 \cdot (3)^6 = 87480$

15. You acquired a baseball card of a famous rookie. A shop owner tells you that it is currently worth 5 dollars and that it will appreciate 20% each year. If you save it for seven years and then give it to your little brother, how much will it be worth?

5 (1.2)
 after 1 year

want a_7 let $a_1 =$ value after 1 year
 $a_7 = a_1 \cdot r^{(n-1)}$
 $a_7 = 5 \cdot (1.2)^6 = 17.92$

OR $5 = a_1$
 100% for 8

16. In order to calculate your grades for this semester, you need to add up the scores on all of your tests. If you scored 65 points on your first test, but raised your score by 7 points each time.

a) Find the total points you have for the six tests this semester.
 $65 + 72 + 79 + 86 + 93 + 100$

OR $S_n = \frac{n(a_1 + a_n)}{2}$
 $= \frac{6(65 + 100)}{2}$
 $= 3(165)$
 $= 495$ points

b) What is the average number of points you earned per test?

$\frac{495}{6} = 82.5$

1. Complete the following formulas.

Arithmetic $a_n = \frac{a_1 + (n-1) \cdot d}{1}$ $S_n = \frac{n(a_1 + a_n)}{2}$

Geometric $a_n = \frac{a_1 \cdot r^{(n-1)}}{1}$ $S_n = \frac{a_1(1-r^n)}{1-r}$

2. Find the next two terms in each sequence. Then decide whether each sequence is arithmetic, geometric, or neither. If possible, identify the common difference or common ratio.

	Next 2 terms	Arithmetic/Geometric/Neither	d/r
a) 2, 6, 18, ...	<u>54, 162</u>	<u>G</u>	<u>r = 3</u>
b) -4, -1, 3, 8, ...	<u> </u>	<u>N</u>	<u> </u>
c) 3, 9, 15, 21, ...	<u>27, 33</u>	<u>A</u>	<u>d = 6</u>
d) 1, 4, 9, 16, ...	<u> </u>	<u>N</u>	<u> </u>
e) -128, 32, -8, ...	<u>2, -1/2</u>	<u>G</u>	<u>r = -1/4</u>

3. Find the indicated term and the general term, a_n .

a) -729, 243, -81, ... $r = -\frac{1}{3}$
 $a_{27} = \frac{-729 \left(-\frac{1}{3}\right)^{26}}{1}$ $a_n = \frac{-729 \left(-\frac{1}{3}\right)^{(n-1)}}{1}$

b) 5.2, 4.5, 3.8, 3.1, ... $d = -0.7$
 $a_9 = \frac{-0.4}{1}$ $a_n = \frac{-0.7n + 5.9}{1}$
 $a_n = 5.2 + (n-1)(-0.7) = -0.7n + 5.9$

c) $\frac{15}{4}, \frac{3}{4}, \frac{3}{20}, \dots$ $r = \frac{1}{5}$
 $a_7 = \frac{15 \cdot \left(\frac{1}{5}\right)^6}{1}$ $a_n = \frac{15 \cdot \left(\frac{1}{5}\right)^{(n-1)}}{1}$

d) -117, -105, -93, ... $d = +12$
 $a_{21} = \frac{123}{1}$ $a_n = \frac{12n - 129}{1}$
 $a_{21} = 12(21) - 129 = 123$ $a_n = -117 + (n-1) \cdot 12 = 12n - 129$

4. Find the indicated values.

a) $a_1 = 3, r = 4, a_7 =$ 1 2 2 8 8

$$a_n = a_1 \cdot r^{n-1}$$

$$a_7 = 3 \cdot 4^6$$

b) $a_6 = 25, a_{15} = 52, a_1 =$ 10 $d =$ 3

6, 25 15, 52

$$\frac{52 - 25}{15 - 6} = \frac{27}{9} = 3 = d$$

$$a_6 = a_1 + (6-1) \cdot 3$$

$$25 = a_1 + 15$$

$a_1 = 10$

e) $a_3 = 64, a_7 = \frac{1024}{81}, r = \pm \frac{2}{3}, a_6 =$ 144

$$\frac{1024}{81} = a_1 \cdot r^6$$

$$64 = a_1 \left(\frac{2}{3}\right)^6$$

$$64 = a_1 \left(\frac{4}{9}\right)^3$$

$$144 = a_1$$

$$\frac{1024}{81} \cdot \frac{1}{64} \cdot \frac{16}{81} = r^4 \quad r = \pm \frac{2}{3}$$

b) $a_1 = 18, d = 5, a_9 =$ 58

$$a_9 = 18 + (9-1) \cdot 5$$

$$= 18 + 8 \cdot 5$$

$$= 58$$

d) $a_5 = -1250, r = -5, a_8 =$ 156250

$$a_n = a_1 \cdot r^{(n-1)}$$

$$-1250 = a_1 \cdot (-5)^4$$

$$\frac{-1250}{625} = a_1$$

$a_1 = -2$

f) $a_{22} = 153, a_{33} = 186, a_1 =$ 90 $d =$ 3

$$d = \frac{186 - 153}{33 - 22} = \frac{33}{11} = 3$$

$$a_n = a_1 + (n-1) \cdot d$$

$$153 = a_1 + (22-1) \cdot 3$$

$$153 = a_1 + 63$$

$$90 = a_1$$

5. Find the indicated means

a) 3 arithmetic means between 24 and 96

a_1 a_5

24 42 60 78 96

$$d = \frac{96 - 24}{5 - 1} = \frac{72}{4} = 18 = d$$

b) 5 geometric means between -16 and -1024

-16 32 -64 -128 -256 -512 -1024

$$-1024 = a_1 \cdot r^6$$

$$\frac{-1024}{-16} = a_1$$

$64 = r^6$

$r = \pm 2$

6. a) In the arithmetic sequence 3, 11, 19, 27, 35, ..., which term is 163?

$a_1 = 3$

$d = 8$

$a_n = 163$

find n

$$a_n = a_1 + (n-1) \cdot d$$

$$163 = 3 + (n-1) \cdot 8$$

$$160 = 8n - 8$$

$$168 = 8n$$

$n = 21$

21st term

b) In the arithmetic sequence $\frac{17}{5}, 4, \frac{23}{5}, \dots$, which term is -14?

$a_1 = \frac{17}{5}$

$a_n = -14$

$d = \frac{3}{5}$

NO SOLUTION!